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ON THE COLLAPSE OF A SPHERICAL CAVITY IN WATER

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1. INTRODUCTION

The phenomenon of cavitation was described by Euler (1)* in 1756 in his theory of turbines. He noted that an insufficient pressure, or even a large pressure in a perfect fluid, can cause disagreement between theory and experiment and can result in a state of zero resistance. Since Euler's introduction to the concept of the formation of cavities in a fluid, the phenomenon has been the subject of a large number of scientific investigations(2).

Cavitation is a dynamic process involving the formation and collapse of vapor-filled cavities in a liquid in motion. In normal liquids these cavities form if the local pressure drops below the vapor pressure. Conversely, a collapse occurs when a cavity is transported into a region of higher pressure. The collapse process can be strongly influenced by the collision between a cavity and a rigid object.

Essentially,there are two types of cavitation. In the first the cavity remains fixed with respect to a bounding surface and usually is observed in hydraulic machinery. As Euler(1) and, later Barnaby(3) noted, an important consequence of this is a decline in machinery efficiency. In fixed cavitation the main flow leaves the guiding surface and follows a free trajectory which usually returns to the surface at some downstream point. The fixed cavity occupies the space between the solid guiding surface and the free liquid surface. Knapp(4) has suggested that this type of cavitation is quite different from the second type, which is characterized by discreet cavities

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* Numbers in parentheses refer to references at the end of the paper.

moving with the fluid. The model for this type of cavitation is the collapse of a spherical bubble.

Even though a typical cavity in water is approximately only one centimeter in diameter, it may collapse violently and generate pressures of the order of hundreds of kilobars. It is not clear which aspect of the collapse is responsible for damage, such as the pitting of propellers. After the collapse a pressure wave is propagated outward in the form of a shock wave, which may cause some damage to neighboring rigid surfaces. On the other hand, the work of Kornfeld and Suvorov(5) indicates that the real damage is done when the cavity strikes the object and not by a pressure wave initiated by a collapse.

To investigate the phenomenon of a cavity collapse completely, account should be taken of the vapor inside the cavity and the effects of viscosity, surface tension, and compressibility. The stability of the process should also be considered. So far, theoretical investigations have treated these properties in isolation (or at best have considered two simultaneously, neither of which was compressibility) and the present work is an extension of considerations which have only dealt with liquid compressibility.

One of the earlier theoretical investigations concerning the collapse of a spherical cavity was made in 1917 by Lord Rayleigh(6). He refers to and extends the work of Besant(7) who calculated the pressure distribution at the time of collapse. Besant's statement of the problem is . . . "An infinite mass of homogeneous incompressible fluid is acted upon by no forces at rest, and a spherical portion of the fluid is suddenly annihilated; it is required to find the instantaneous alteration of pressure at any point of the mass, and the time in which the cavity will be filled up, the pressure at an infinite distance supposed to remain constant."

Rayleigh integrates the equation of continuity to obtain the velocity distribution for the flow. He derives the following expression for the velocity,

$$u = \frac{R^2 \dot{R}}{r^2} , \tag{1.1}$$

where dots indicate differentiation with respect to time, R is the cavity radius, and r is the radial coordinate. For the motion of the cavity wall, with the use of the principle of conservation of total energy, he obtains

$$R \ddot{R} + \frac{3}{2} \dot{R}^2 = \frac{-p_o}{\rho} , \tag{1.2}$$

where p_o is the pressure in the fluid at rest and ρ is the density. Lord Rayleigh also derives for the pressure that

$$\frac{p}{p_o} = \frac{R_o^{\,2}}{4^{4/3} R^3}. \tag{1.3}$$

From Equation 1.2 $\dot{R} = U = -\infty$ when R = 0. Equation 1.3 shows that when R = 0, the pressure p is infinite for finite values of r. Therefore, the energy distribution is infinite and singular at that instant. Rayleigh also shows that if the original cavity radius is R_o, the collapse time is proportional to

$$R_o \rho^{1/2} \, p^{-1/2} \tag{1.4}$$

and is therefore always finite. Equation 1.4 follows directly from dimensional considerations.

Rayleigh realized that compressibility should be considered in the later stages of high speed collapse, and referred to work by Cook who took account of the compressibility of the cavity contents (he assumed that the cavity was not empty and that the contents obeyed Boyle's law) but not of the surrounding liquid. Cook examined the mode of collapse when the cavity strikes a rigid sphere. His approximation yields plausible pressures (68 tons per square inch in a particular example) and predicts that the collapse velocity will be zero before the cavity reaches its minimum radius, i.e., that the collapse time is infinite. However, since his work does not account for the compressibility of the surrounding media, it is not satisfactory from a theoretical point of view.

Investigations concerning the roles of surface tension and viscosity have been carried out by Poritsky(8) and also by Shu(9). Poritsky discusses the collapse or growth of a spherical cavity in an incompressible viscous fluid and he points out and resolves the following interesting paradox concerning the role of viscosity. According to the equation of continuity for the motion of an incompressible fluid with spherical symmetry, the velocity $\vec{V}$ is given by

$$\vec{V} = \left(\frac{c(t)}{r^2}\right)\vec{e}_r \,, \tag{1.5}$$

where c(t) is an admissible time function and $\vec{e}_r$ is a unit vector along the radius r. Hence we can define a velocity potential ϕ as follows,

$$\vec{V} = \nabla\phi \,, \tag{1.6}$$

so that

$$\phi = -\frac{c(t)}{r} \,. \tag{1.7}$$

Since $\nabla^2\phi = 0$,

$$\nabla^2\vec{v} = \nabla^2(\nabla\phi) = \nabla(\nabla^2\phi) = 0 \tag{1.8}$$

so that the viscosity term in the Navier Stokes equations

$$\rho\frac{D\vec{V}}{Dt} = \vec{F} - \text{grad } p + \mu\nabla^2\vec{V}, \tag{1.9}$$

vanishes and they reduce to the Eulerian equations of motion.

Poritsky resolves the paradox by noting that while it is true that the effect of viscosity vanishes in the equations of motion, so that the resultant viscous stress per unit volume at any point in the fluid domain vanishes, this is not necessarily the case with the stresses themselves. At any point the three principle stresses p_i and strain rates e_i are given by

$$p_i = -p - \frac{2}{3}\mu(e_1 + e_2 + e_3) + 2\mu e_i, \tag{1.10}$$

$(i = 1,2,3)$. But for an incompressible fluid,

$$e_1 + e_2 + e_3 = 0, \tag{1.11}$$

and hence we may calculate the pressure at the cavity wall from Equation 1.10. We may not calculate the pressure from the pressure in the fluid at the cavity wall, and if p is this fluid pressure, then for the pressure p_o at the cavity we have

$$p_o = p - 2\mu e. \tag{1.12}$$

Here μ is the coefficient of viscosity and e is one of the components e_i. As Poritsky emphasized, viscosity only enters the problem in this boundary condition.

Poritsky is mainly concerned with the kinematics of the collapse and he does not calculate the velocity or pressure fields. It is apparent that if compressibility is accounted for, then the role of viscosity remains in the Navier-Stokes equations. Also, we no longer have the condition imposed by isochoric motion, i.e., Equation 1.11.

Poritsky finds that

$$\frac{p_o - p_\infty}{\rho} = \ddot{R}R + \frac{3}{2}(\dot{R})^2 + \frac{4\mu}{\rho}\frac{\dot{R}}{R}, \tag{1.13}$$

where p_o is given by Equation 1.12 and p_∞ is the pressure in the fluid at rest. The modifications of Equation 1.13 to admit the effect of surface tension is

$$\frac{p_o - \frac{2\sigma}{R} - p_\infty}{\rho} = R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{4\mu}{\rho}\frac{\dot{R}}{R}. \tag{1.14}$$

where σ is the surface tension constant.

He finds that the collapse time is infinite for some values of the coefficient of viscosity and Shu establishes rigorously (with modifications of the Poincaré-Bendixon theory of non-linear differential equations) that the time of collapse is infinite if a non-dimensional viscosity c is greater than a critical value $c_o(<\sqrt{6})$ and is finite otherwise. Poritsky shows by means of a numerical example that if surface tension is accounted for, then the collapse time is finite and Shu claims that this can be shown to always be true. Poritsky also shows that surface tension speeds up the collapse. It should be noted that Rayleigh's theory always predicts finite collapse times.

Zwick and Plesset(10) investigated the dynamics of small vapor bubbles in an incompressible, inviscid liquid. Their main conclusions may be summarized as follows. The vapor pressure at the cavity wall is determined by the temperature there. Due to the latent heat required for evaporation, a change in bubble size will lead to heat transfer across the bubble wall causing the surrounding liquid to heat when the bubble gets smaller and to cool during bubble growth. Heating the liquid causes an increase in the vapor pressure and hence slows down the collapse.

The results obtained by Zwick and Plesset show that, even though the temperature of the bubble wall increases rapidly during the later stages of collapse, the motion of the cavity wall agrees very closely with that predicted by Rayleigh. Hunter(11) concludes from these results that the final collapse may take place too rapidly for the vapor to condense and that the vapor cavity may collapse as an empty cavity until the vapor pressure becomes sufficiently large to cause the cavity wall to rebound.

Plesset(12) has derived the equation of motion for the bubble radius R in an inviscid, incompressible fluid. He finds that

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{(p(R) - Po)}{\rho}, \tag{1.15}$$

where ρ is the liquid density, Po is the external pressure at $r = \infty$, and $p(R)$ is the pressure in the liquid at the bubble boundary and is given by

$$p(R) = p_v(T) - 2\sigma/R. \tag{1.16}$$

Here $p_v(T)$ is the equilibrium vapor pressure corresponding to temperature T at the bubble boundary.

The effect of compressibility on the collapse of a spherical bubble has been considered by Gilmore(13), Brand(19) and Hunter(11). Brand(19) integrates the Eulerian equations of motion numerically using the method of characteristics and compares results with those obtained by solving the Lagrangian equations in series. A Tait equation of state is used, and viscosity and surface tension are neglected. Hunter also carries out a calculation by the method of characteristics but continues up to the instant of final collapse, while Brand's results stop short of this.

Hunter noted from his numerical results that the motion in the neighborhood of the collapse is self similar, to a high approximation. However, in formulating his similarity theory, Hunter is forced to approximate vacuum conditions at the cavity wall.

In Hunter's theory there is, in contrast to Rayleigh's theory, no singular energy distribution at the time of collapse. The similarity solution predicts an infinite velocity at the time of collapse but gives finite pressures for all radial distances at this time. Hunter shows that compressibility retards the collapse process.

The collapse of a cavity in an incompressible fluid has been shown to be unstable by Birkhoff (14).

The present paper is essentially an extension of the work of Hunter. The existence of a self-similar hypothesis is shown to be implied by dimensional arguments. Hunter's zero order boundary condition, that the sound speed is zero at the cavity wall, is corrected by use of a perturbation scheme which is developed in the text. Also, several of Hunter's results which rely on energy considerations are more suitably obtained here. It is also shown that the cavity wall velocity is decreased when account is taken of the non-zero sound speed at the cavity wall. This collapse velocity correction is obtained without numerical integration of the equations governing the first approximation to the non-self-similar motion. Since this result is the main object of the paper, the perturbation equations are not integrated here although the procedure for doing this is described in full.

2. FORMULATION OF THE PROBLEM

(1) Similarity and Dimensional Techniques. Analysis of the Primitive Equations.

The symmetrical collapse of a cavity in water is an example of unsteady motion of a compressible fluid with spherical symmetry. Under certain conditions, as shown by Taylor(20), Sedov(15), Stanyukovich(16) Guderley(21) and other authors, the problem may be investigated by similarity techniques. We shall discuss the concept of self-similar motion, and then investigate the possible self-similar characteristics of the collapse problem.

A motion is said to be self-similar if the spatial distribution of the flow parameters at a certain instant is identical with that at any other time, apart from a change in scale.

The unsteady motion of a gas with plane, cylindrical or spherical symmetry, obeying a certain type of equation of state, can be shown to be self-similar provided that the number of constants which arise with independent dimensions does not exceed two. For completeness we shall include the discussion of this point given by Sedov. Much of this is relevant to Hunter's analysis of the cavity collapse problem.

Let us analyse the dependent variables and fundamental parameters arising in unsteady motion in one space coordinates. In the Eulerian formulation, the physical variables may be taken as the velocity v, the density ρ, and the pressure p. The characteristic parameters are the radial distance r, the time t, the dimensional constants which enter into the problem, and the boundary and initial conditions.

Since the density and pressure have dimensions which contain the mass, at least one constant a, must have mass occurring in its dimensions. We can assume that the dimensions of a are

$$[a] = ML^kT^s. \tag{2.1}$$

Then we may write for the velocity, density, and pressure

$$v = \frac{r}{t}V, \quad \rho = \frac{a}{r^{k+3}t^s}R, \quad p = \frac{a}{r^{k+1}t^{s+2}}P, \tag{2.2}$$

where V, R and P are arbitrary and can depend on non-dimensional combinations of r, t and other parameters.

But if another constant b entering the problem has dimensions independent of a, and all other constants have dimensions depending on a and b only, then we may write

$$[b] = L^mT^n. \tag{2.3}$$

Now the number of independent variables that can be formed by combination of a and b is reduced to one, and the variables v, ρ, p will depend on this one non-dimensional variable. Moreover, this variable will be

$$\frac{r^m t^n}{b}, \tag{2.4}$$

and if $m \neq o$, this similarity variable is equivalent to a variable

$$\lambda = \frac{r}{b^{1/m}\, t^{\delta}}, \text{ where } \delta = -\frac{n}{m}. \tag{2.5}$$

Hence we have shown that if there are only two dimensional constants with independent dimensions, then the motion is self-similar. We now investigate the equations governing the collapse problem and determine the number of constants with independent dimensions.

(2) The Role of the Energy

It is useful to clarify the role of the energy in the present problem. The total energy of the flow is given by

$$\int_0^\infty \rho[\mathcal{E} + \tfrac{1}{2} u^2] 4\pi r^2 dr, \qquad (2.6)$$

where $\mathcal{E}$ is the internal energy and u is the velocity. It will be seen later that the similarity solution to the present problem is valid within a small spherical domain (approximately one centimeter in diameter). Hence energy considerations are strictly inapplicable in determining a lower bound for the similarity parameter, since the functions u and $\mathcal{E}$ given by the similarity solution cannot be used to compute the relation given by Eq. 2.6 . But if the velocity distribution and expressions for $\mathcal{E}$ are derived by integrating the governing partial differential equations exactly, then we can impose the condition that energy integral (Equation 2.6) should converge uniformly.

(3) The Model

We now describe the model of collapse and the dimensional consequences of this model. Following Hunter (see footnote, ibid. pg. 246) we shall consider a cavity initially of infinite size which has been collapsing for an infinite amount of time. As he points out, the motiviation for this model comes from the incompressible treatment. For from conservation of total energy (using Rayleigh's incompressible treatment) we have

$$\dot{R}^2 = \frac{2p_0}{3\rho}\left(\frac{R_0^3}{R^3} - 1\right),$$

and for $R_o >> R$, we see that

$$\dot{R}^2 \doteq \frac{2p_oR_o^3}{3\rho R^3} . \qquad (2.7)$$

Therefore the motion does not depend on the scales p_o and R_o separately but on their combination and since $p_oR_o^3$ is equal to $\frac{3E}{4\pi}$, where E is the initial energy of the system, and if we let $p_o \to o$ and $R_o \to \infty$ in such a way that E remains constant, then the flow is formally determined by the parameter E alone. We are now in a position to formulate the problem.

(4) Equations of Motion, Boundary Conditions, Initial Conditions

The momentum equation and the continuity equation corresponding to unsteady motion of a gas with spherical symmetry are respectively:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial r'} + \frac{1}{\rho'}\frac{\partial p'}{\partial r'} = 0 \qquad (2.8)$$

$$\frac{\partial \rho'}{\partial t'} + u'\frac{\partial \rho'}{\partial r'} + \rho'\left(\frac{\partial u'}{\partial r'} + \frac{2u'}{r'}\right) = 0 . \qquad (2.9)$$

where the primes are used for dimensional quantities.

The Tait equation of state for water(22) may be written in the form

$$\frac{p' + B}{B} = \left(\frac{\rho'}{\rho_o}\right)^{\gamma} . \qquad (2.10)$$

Here B and ρ_o are slowly varying functions of entropy and they may be assumed to be constant; γ is taken equal to 7 in the future calculations. The velocity of sound c' is given by

$$c'^2 = \left(\frac{dp'}{d\rho'}\right)_s = \frac{\gamma B\rho'^{\gamma-1}}{\rho_o^{\gamma}}, \qquad (2.11)$$

and if $c' = c_o$ when $\rho' = \rho_o$, then

$$\rho_o c_o^2 = \gamma B. \tag{2.12}$$

Following Hunter we shall use the three dimensional scales E, ρ_o, c_o to express all variables in dimensionless form. We then have (where variables without dashes are non-dimensional)

$$\left.\begin{aligned} &u' = u c_o,\ c' = c\, c_o,\ \rho' = \rho\, \rho_o,\ r' = r\left[\frac{5}{2c_o}\sqrt{\frac{E}{2\pi\rho_o}}\right]^{3/2}, \\ &t' = \frac{t}{c_o}\left[\frac{5}{2c_o}\sqrt{\frac{E}{2\pi\rho_o}}\right]^{2/3},\ \mathcal{E} = \mathcal{E} c_o^2, \end{aligned}\right\} \tag{2.13}$$

where the unprimed quantities are dimensionless.

Then, using Equations 2.10, 2.11, 2.12, and 2.13, Equations 2.8 and 2.9 may be written:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\gamma - 1}\frac{\partial c^2}{\partial r} = 0 \tag{2.14}$$

$$\frac{\partial c^2}{\partial t} + u\frac{\partial c^2}{\partial r} + (\gamma-1)c^2\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) = 0 \tag{2.15}$$

The boundary conditions are

$$c = 1 \text{ and } u = \dot{R} \text{ at the cavity wall, } r = R \tag{2.16}$$

and

$$c \to 1 \text{ as } r \to \infty \tag{2.17}$$

We observe that in the incompressible treatment condition (Equation 2.16) would determine the motion. The effect of compressibility is felt through the initial conditions and hence the spatial variables must have an assigned distribution at a given time. Hunter uses the results of Rayleigh's incompressible treatment to derive initial conditions. From Equations 1.1, 1.2, and 2.7 it follows that

$$\left.\begin{array}{l} R \sim -t^{2/5}; \quad u \sim + \dfrac{2(t)^{1/5}}{5r^2} \\ c^2 = 1 + \dfrac{2(\gamma-1)(-t)^{-6/5}}{25} \left[\dfrac{(-t)^{2/5}}{r} + \dfrac{(+t)^{8/5}}{r^4} \right] \end{array} \right\} \text{ as } -t \to \infty \qquad (2.18)$$

These initial conditions are specified as $t \to -\infty$, when the fluid can be regarded as incompressible. Hence the realistic conditions in the final part of the collapse have been preserved and the Equations 2.14, 2.15, 2.16, 2.17 and 2.18 completely specify the flow.

The governing Equations 2.8 and 2.9, together with the associated boundary and initial conditions may be replaced by the equivalent system Equations 2.13, 2.14, 2.15, 2.16, 2.17 and 2.18 . We then see that three constants with independent dimensions arise, namely, c_o, ρ_o, and E. To satisfy the requirements of similarity, one of these must be eliminated. Hunter puts c_o equal to zero, which means that vacuum conditions apply at the cavity wall. Hunter's justification for this step is that, although the dimensionless sound speed at the cavity wall is in fact equal to unity, in the region where similarity solution applies $c^2 >> 1$.

In the present analysis we take account of the fixed sound speed at the cavity wall; (note that Equation 2.12 shows that $c_o = 0$ implies $B = 0$ and hence a perfect gas law for water) our development will be a perturbation of Hunter's similarity solution such that his solution is the zeroth approximation of our perturbation scheme.

3. CAVITY COLLAPSE WITH A NON-ZERO SOUND SPEED AT THE CAVITY WALL

(1) Perturbation Theory

In the present problem three dimensional constants enter so that the similarity hypothesis is violated. Several investigators have dealt with similar situations, particularly in explosion problems. Sedov(15) discusses the point explosion while taking counter pressure into account. A similar technique was employed by Sakurai(17) in the problem of a blast wave propagation with counter pressure. The analysis given here is similar to that of Sakurai.

We define

$$u = \dot{R}_H f(x,y), \quad c^2 = \dot{R}_H^2 g(x,y) \tag{3.1}$$

$$\text{where} \quad x = \frac{r}{R} \quad \text{and} \quad y = \frac{c|_{r=R}}{\dot{R}_H} = \frac{1}{\dot{R}_H}; \tag{3.2}$$

u is the fluid velocity, $\dot{R}_H$ is the velocity of the cavity wall as given in Hunter's treatment (i.e. from self-similar equations), and c is the sound speed. Here x is the same parameter used by Hunter in his self-similar solution and y is our perturbation variable. All quantities are dimensionless. We shall expand f and g in powers of y^2 as follows:

$$f = f^{(o)} + y^2 f^{(1)} + y^4 f^{(2)} + \ldots . \tag{3.3}$$

$$g = g^{(o)} + y^2 g^{(1)} + y^4 g^{(2)} + \ldots . \tag{3.4}$$

where $f^{(i)}$ and $g^{(i)}$, $i = 0, 1, \ldots, n$ so are assumed to be functions of x alone.

We are assuming that Hunter's solution is a valid zeroth approximation and hence we accept his determination of a similarity index n defined by

$$\frac{\ddot{R}R}{\dot{R}_H^2} = 1 - \frac{1}{n} = \mathcal{H}, \text{ a constant.} \tag{3.5}$$

Hunter finds that n has the value .5552, and a complete discussion of his method may be found in the appendix. (It is noteworthy that the use of Equation 3.5 simplifies the analysis to follow; namely, the derivation of the non-self similar equations to determine $f^{(i)}$, $g^{(i)}$ for $i = 1, 2, \ldots, n$.)

By substitution of Equations 3.1 and 3.5 in Equations 2.14 and 2.15, and by use of the expressions

$$\left.\begin{aligned} \frac{\partial}{\partial r} &= \frac{1}{R}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} &= -\frac{x\dot{R}}{R} - \frac{\mathcal{H}}{R}\frac{\partial}{\partial y} \end{aligned}\right\} , \tag{3.7}$$

we obtain, after some reduction

$$-x\frac{\partial f}{\partial x} + 2\mathcal{H}f - \mathcal{H}y\frac{\partial f}{\partial y} + f\frac{\partial f}{\partial x} - \frac{f^2}{x}\mathcal{H} + \frac{1}{\gamma-1}\left(\frac{\partial g}{\partial x} - \frac{2\mathcal{H}g}{x}\right) = 0, \tag{3.8}$$

and

$$-x\frac{\partial g}{\partial x} + 4\mathcal{H}g - \mathcal{H}y\frac{\partial g}{\partial y} + f\frac{\partial g}{\partial x} - \frac{2fg\mathcal{H}}{x} + g(\gamma-1)\left(\frac{2f}{x} + f_x - \frac{f\mathcal{H}}{x}\right) = 0 \tag{3.9}$$

Substituting Equations 3.3 and 3.4 into Equations 3.8 and 3.9 respectively and comparing coefficients of unity and y^2 we obtain the zero and first order approximations. The equations may be arranged as follows:

<u>Zero order approximation</u>

$$\left[g^{(o)} - (x-f^{(o)})^2\right]\frac{df^{(o)}}{dx} = g^{(o)}\left(-\frac{2\mathcal{H}}{\gamma-1}\right) - 2\mathcal{H}xf^{(o)} + f^{(o)2}(3\mathcal{H}) - \frac{f^{(o)3}}{x}\mathcal{H} + f^{(o)}g^{(o)}\left(\frac{\mathcal{H}-2}{x}\right) \tag{3.10}$$

$$\left[g^{(o)} - (x-f^{(o)})^2\right]\frac{dg^{(o)}}{dx} = -4\mathcal{H}g^{(o)}x + f^{(o)}g^{(o)}\left[-\mathcal{H}(\gamma-1) + 2\mathcal{H} - 2(\gamma-1)\right] + g^{(o)2}\left(\frac{2\mathcal{H}}{x}\right) + f^{(o)2}g^{(o)}\left[\frac{2(\gamma-1) - 2\mathcal{H}}{x}\right] \tag{3.11}$$

First order approximation

$$\left[(f^{(o)}-x)^2-g^{(o)}\right]\frac{df^{(1)}}{dx}=g^{(1)}\beta_1(x)+f^{(1)}\beta_2(x) \qquad (3.12)$$

$$\left[(f^{(o)}-x)^2-g^{(o)}\right]\frac{dg^{(1)}}{dx}=g^{(1)}\beta_3(x)+f^{(1)}\beta_4(x). \qquad (3.13)$$

Here

$$\beta_1=\frac{2f^{(o)}}{x}+f_x^{(o)}-\frac{\mathcal{H}f^{(o)}}{x}$$

$$\beta_2=xf_x^{(o)}-2\mathcal{H}f^{(o)}-f^{(o)}f_x^{(o)}+\frac{2\mathcal{H}}{x}f^{(o)2}+\frac{1}{\gamma-1}g_x^{(o)}+g^{(o)}\left[\frac{2}{x}-\frac{2\mathcal{H}}{x(\gamma-1)}-\frac{\mathcal{H}}{x}\right],$$

$$\beta_3=2\mathcal{H}x-f^{(o)}f_x^{(o)}(\gamma-1)+xf_x^{(o)}(\gamma-1)+f^{(o)}\left(2(\gamma-1)-\mathcal{H}(\gamma-1)-4\mathcal{H}\right)-\frac{2\mathcal{H}g^{(o)}}{x}+f^{(o)2}\left(-\frac{2(\gamma-1)}{x}+\frac{\mathcal{H}(\gamma-1)}{x}+\frac{2\mathcal{H}}{x}\right), \qquad (3.14)$$

$$\beta_4=xf_x^{(o)}-2\mathcal{H}f^{(o)}-f^{(o)}f_x^{(o)}+\frac{2\mathcal{H}}{x}f^{(o)2}+\frac{1}{\gamma-1}g_x^{(o)}+g^{(o)}\left[\frac{2}{x}-\frac{2\mathcal{H}}{x(\gamma-1)}-\frac{\mathcal{H}}{x}\right].$$

(2) Boundary Conditions

As we have seen, the exact boundary conditions are

$$u=\dot{R} \text{ at } r=R \qquad (3.15)$$

and

$$c=1 \text{ at } r=R\,. \qquad (3.16)$$

Let us first consider the zero order approximation . Then we have

$$u=\dot{R}_H f(x,0)=\dot{R}_H f^{(o)}(x), \qquad (3.17)$$

and

$$c^2=\dot{R}_H^2 g(x,0)=\dot{R}_H^2 g^{(o)}(x). \qquad (3.18)$$

The functions $f^{(o)}$ and $g^{(o)}$ are solutions of Equations 3.10 and 3.11. They are precisely the functions that occur in Hunter's treatment. We see that we are unable to satisfy the exact boundary condition (Equation 3.16) in the zero order solution (Equation 3.18), and hence to this degree of approximation we have $g^{(o)}(1) = 0$. Obviously condition (Equation 3.15) is satisfied by setting $f^{(o)}(1) = 1$.

We now substitute the complete expansions (Equations 3.3 and 3.4) into Equation 3.1 and apply the boundary conditions given exactly by Equations 3.15 and 3.16, namely,

$$u\Big|_{r=R} = \dot{R}_H f(1,\frac{1}{\dot{R}_H}) = \dot{R}_H \left[f^{(o)}(1) + \frac{1}{\dot{R}_H^2} f^{(1)}(1) + \ldots\right] = \dot{R}, \tag{3.19}$$

and

$$c^2\Big|_{r=R} = \dot{R}_H^2 g(1,\frac{1}{\dot{R}_H}) = \dot{R}_H^2 \left[g^{(o)}(1) + \frac{1}{\dot{R}_H^2} g^{(1)}(1) + \ldots\right] = 1 \tag{3.20}$$

Therefore, if we set

$$f^{(i)}(1) = \beta^{(i)} \text{ for } i = 1,2,\ldots n, \text{ where } \beta^{(i)} \text{ is to be} \tag{3.21}$$

determined, and

$$g^{(1)}(1) = 1,\ g^{(i)}(1) = 0 \text{ for } i = 2,\ 3,\ \ldots n, \tag{3.22}$$

then we can satisfy the boundary conditions exactly.

Therefore, the boundary conditions for the similarity Equations 3.10 and 3.11 are:

$$f^{(o)}(1) = 1, \tag{3.23}$$

and

$$g^{(o)}(1) = 0. \tag{3.24}$$

For the first perturbation of the self similar solution we have

$$f^{(1)}(1) = \beta^{(1)} \tag{3.25}$$

$$g^{(1)}(1) = 1. \tag{3.26}$$

We shall now discuss the zero order equations, closely following Hunter's development. Since Hunter does not tabulate the functions $f^{(o)}$, $g^{(o)}$ it is necessary to integrate Equations 3.10 and 3.11. In the following discussion we shall also show that these solutions can be fixed without appeal to the energy considerations used by Hunter. Further, Hunter's discussions of the singularities of Equations 3.10 and 3.11 will be clarified.

4. THE SELF SIMILAR EQUATIONS

In the appendix a complete discussion of the following items is given:

(a) The transformation theory employed by Hunter in his solution of the self similar equations.

(b) A discussion of the singular points of the transformed equations.

(c) The application of techniques given by Stanyukovich(16) which yield the same conclusions as those obtained by Hunter but which do not require the use of energy arguments.

(d) A discussion and results of the numerical integration of the transformed equations. (Hunter gives the solution curves but he does not tabulate the functions $f^{(o)}$ and $g^{(o)}$. Since they occur in the coefficients of the non-self similar equations it was necessary to repeat the integration).

In Figure 1 we give the results of the integration of the self similar equations; Table 1 is a tabulation of the function $f^{(o)}$, $g^{(o)}$. It is important to note that the parabola $(f^{(o)} - x)^2 = g^{(o)}$ locates the singular points of the $f^{(o)}$; $g^{(o)}$ and $f^{(1)}$, $g^{(1)}$ equations.

5. ANALYSIS OF THE NON-SELF-SIMILAR EQUATIONS

The non-self-similar motion is described by Equations 3.12 and 3.13. These equations are linear differential equations with non-constant coefficients having singular points at the same location as the non-linear Equations 3.10 and 3.11; namely, at the two points where the quantity $g^{(o)} - (x - f^{(o)})^2$ vanishes. Obviously this vanishes at the cavity wall $x = 1$, and the numerical integration of the non-linear equations shows that the expression also vanishes at $x = 1.51$. The following expansions are immediately obtained from the non-linear Equations 3.10 and 3.11.

Near the cavity wall $x = 1$.

$$\left.\begin{aligned} f^{(o)} &= 1 - 2.0292(x-1) + \ldots \\ g^{(o)} &= 4.8066\,(x-1) + \ldots \end{aligned}\right\} \tag{5.1}$$

and near the singular point $x = 1.51$,

$$\left.\begin{aligned} f^{(o)} &= .58 - .57066\,(x-1.51) + \ldots \\ g^{(o)} &= .86 + .409223\,(x-1.51) + \ldots \end{aligned}\right\} \tag{5.2}$$

When the expressions (Equation 5.1) are used in Equations 3.12 and 3.13, the following expansions are found for $f^{(1)}$ and $g^{(1)}$; here the coefficient B is arbitrary and must be determined from uniqueness considerations. We have

$$g^{(1)} = 1 - 11.354(x-1) + B\left((x-1)^{7/6} - 2.979(x-1)^{13/6} + \ldots \right) \tag{5.3}$$

and

$$\begin{aligned} f^{(1)} = &\left\{\frac{-4.807 + 4.369(x-1) + 9.176(x-1)^2}{-28.165 + 130.309\,(x-1)}\right\} \Big\{ -11.359 + \\ &B\left(1.166(x-1)^{1/6} - 6.454(x-1)^{7/6} + \ldots\right)\Big\} \\ &- \left\{\frac{28.326 - 116.47(x-1)}{-28.165 + 130.309(x-1)}\right\} \Big\{ 1 - 11.359(x-1) \\ &+ B\left((x-1)^{7/6} - 2.979(x-1)^{13/6} + \ldots \right)\Big\} \end{aligned} \tag{5.4}$$

(1) Analysis at the Cavity Wall

We now consider the boundary conditions at the cavity wall. The velocity is given by

$$u = \dot{R}_H \left(f^{(o)} + \frac{1}{\dot{R}_H^2} f^{(1)} + \ldots\right),$$

and at the cavity wall

$$\dot{R} = \dot{R}_H \left(f^{(o)} + \frac{1}{\dot{R}_H^{\,2}} f^{(1)} + \ldots\right),$$

or with

$$\dot{R} = \dot{R}_H + \mathcal{E}^{(1)}, \tag{5.5}$$

where $\mathcal{E}^{(1)}$ is the correction to the self similar theory,

$$\dot{R}_H + \mathcal{E}^{(1)} = \dot{R}_H\left(1 + \frac{1}{\dot{R}_H^{\,2}} f^{(1)}(1)\right). \tag{5.6}$$

From Equation 5.4 we have at x = 1,

$$f^{(1)} = -.96367, \tag{5.7}$$

and hence Equation 5.6 becomes

$$\mathcal{E}^{(1)} = -\frac{.96367}{\dot{R}_H}, \tag{5.8}$$

and therefore the new velocity of the cavity wall is given by

$$\dot{R}^{(1)} = \dot{R}_H - \frac{.96367}{\dot{R}_H} \tag{5.9}$$

The comparison between $\dot{R}$ and $\dot{R}_H$ is shown in Figure 2.

(2) Analysis at the Singular Point x = 1.51

With the use of expressions (Equation 5.2), Equations 3.12 and 3.13 have the following form in the neighborhood of the singular point x = 1.51:

$$\left[c_1 (x-1.51) + c_2 (x-1.51)^2 + c_3 (x-1.51)^3\right] \frac{dg^{(1)}}{dx} \tag{5.10}$$

$$= \left[\ell_1 + \ell_2(x-1.51) + \ell_3(x-1.51)^2\right] g^{(1)} + \left[p_1 + p_2(x-1.51)+p_3(x-1.51)^2\right] f^{(1)},$$

$$\left[c_1(x-1.51) + c_2(x-1.51)^2 + c_3(x-1.51)^3\right] \frac{df^{(1)}}{dx}$$

$$= \left[q_1 + q_2(x-1.51)\right] g^{(1)} + \left[r_1 + r_2(x-1.51) + r_3(x-1.51)^2\right] f^{(1)}. \tag{5.11}$$

Here

$\ell_1 = 5.5178$	$q_1 = 1.94511$
$\ell_2 = 9.0724$	$q_2 = -2.16913$
$\ell_3 = -24.3944$	$r_1 = .48064$
$p_1 = .31514$	$r_2 = 2.1183$
$p_2 = 6.8397$	$r_3 = -2.3324$
$p_3 = 8.8287$	$c_1 = 3.7933$
$c_2 = 6.2369$	$c_3 = 2.4668$

At x=1.51 expansions for $f^{(1)}$ and $g^{(1)}$ given by Equations 5.10 and 5.11 show that

$$f^{(1)} \sim (x-1.51)^{\alpha_1} \tag{5.12}$$

$$g^{(1)} \sim (x-1.51)^{\alpha_2}, \tag{5.13}$$

where α_1 and α_2 have positive non-integral values. In fact the second order equation which can be derived from Equations 5.10 and 5.11 has an indicial equation with roots 2.79 and .374. Therefore, apart from the solution $f^{(1)} = g^{(1)} = 0$, there are no regular integrals at the singular point $x = 1.51$. It is also important to note that if one function at $x = 1.51$ is specified the other is fixed, since for example,

$$\left\{p_1 + p_2(x-1.51) + p_3(x-1.51)^2\right\} f^{(1)}$$
$$= \left\{c_1(x-1.51) + c_2(x-1.51)^2 + c_3(x-1.51)^3\right\} g^{(1)}{}_x +$$
$$\left\{\ell_1 + \ell_2(x-1.51) + \ell_3(x-1.51)^2\right\} g^{(1)}, \tag{5.14}$$

and there exists a similar expression for $g^{(1)}$ in terms of $f^{(1)}$ and $f^{(1)}{}_x$.

Hence we have the following situation. There are two singular points in the field. The expansions, Equation 5.3 and 5.4, describe the behavior of $f^{(1)}$ and $g^{(1)}$ near the cavity wall singularity and also $f^{(1)} = -.96367$ and $g^{(1)} = 1.0$ there, but there is an arbitrary constant B in the expansions. At the singular point $x = 1.51$ there are an infinity of irregular integrals.

Let us consider the asymptotic behavior of solutions of the perturbation equations. For large values of x, the β's in Equations 3.12 and 3.13 become

$$\left.\begin{aligned} \beta_1 &\sim 2Mx^{-.3} \\ \beta_2 &\sim -(2\gamma M + 1.3M)x^{-1.3} \\ \beta_3 &\sim 2\gamma x + (20-7M)x^{-1.3} \\ \beta_4 &\sim -x^{-1.3}(2\gamma M + M) \end{aligned}\right\} \tag{5.15}$$

where $o \leq M \leq 1$ and in fact $M \to o$ as x increases. Therefore, from Equations 3.12 and 3.13 we see that the asymptotic behavior of $f^{(1)}$ and $g^{(1)}$ is defined by

$$\left.\begin{aligned} f^{(1)} &\sim -\frac{2}{3} MC_1 x^{-3} \\ g^{(1)} &\sim C_2 x^{-1.602} \end{aligned}\right\} \tag{5.16}$$

where c_1 and c_2 are arbitrary constants.

Equation 5.16 shows that all integral curves define perturbation variables which tend to zero at infinity so that the similarity solution is approached as the distance from the cavity wall is increased.

To find the value of the constant B and the pair of integral curves through $x = 1.51$ a further condition between the fluid velocity and velocity of sound must be satisfied in the finite part of the field. This can be derived from the characteristics solutions of Hunter or Brand.

It should be emphasized once more that the most useful product of the perturbation scheme is the correction to the cavity wall velocity, and this is found without integrating the governing equations throughout the field.

APPENDIX

(1) The Self Similar Equations

Following Hunter, we shall apply the following transformations to Equations 3.10 and 3.11. Let

$$\left.\begin{aligned} x &= -\xi^{1/n}, \quad F(\xi) = x^{(1/n)-1} f^{(o)}(x), \\ G(\xi) &= x^{(2/n)-2} g^{(o)}(x), \end{aligned}\right\} \tag{A-1}$$

where $n = .5552$.

Let us further transform according to

$$X = \log(-\xi), \quad Y = -\xi F, \quad Z = \xi^2 G. \tag{A-2}$$

Then we may arrange the transformed equations into the form

$$dX : dY : dZ = (Y-1)^2 - Z : \left[Y(Y-1)(nY-1) - \frac{1}{3} Z(9nY + n-1)\right]$$
$$: 2Z\left[-nZ + 7nY^2 + 2Y(1-5n) + 1\right]. \tag{A-3}$$

Let us consider the equation for $\frac{dY}{dZ}$ and for convenience for later discussion define $y = Y - y_o$, $z = Z - z_o$. Then from (A-3) we have

$$\frac{dy}{dz} = \frac{a_{1o}z + a_{o1}y + P(y,z)}{b_{1o}z + b_{o1}y + Q(y,z)}, \tag{A-4}$$

where

$$P(y,z) = 3nyz + y^2(-n-1 + 3ny_o) + ny^3, \tag{A-5}$$

and

$$Q(y,z) = yz(4-20n + 28ny_o) + z^2(-2n)$$
$$+ y^2(14nz_o) + zy^2(14nz_o). \tag{A-6}$$

Also,

$$a_{10} = \frac{1-n}{3} - 3ny_o,$$

$$a_{01} = 1-3nz_o - 2(n+1)y_o + 3ny_o^2, \qquad \text{(A-7)}$$

$$b_{10} = 2 + y_o(4-20n) - 4nz_o + 14ny_o^2,$$

and

$$b_{01} = z_o(4-20n) + 28ny_oz_o.$$

Now if we let

$$y = vz, \qquad \text{(A-8)}$$

then equation (A-4) becomes

$$z\frac{dv}{dz} = \frac{a_{10} + a_{01}v - b_{10}v - b_{01}v^2 + z\,\psi_1(v,z),}{b_{10} + b_{01}v + z\,\psi_2(v,z).} \qquad \text{(A-9)}$$

where

$$\psi_1 = (-1-n + 3ny_o)v^2 - 3nv + nv^3z - 14nz_ov^3$$
$$+ 2nv - (4-20n + 28ny_o)v^2 - 14nv^3z, \qquad \text{(A-10)}$$

and

$$\psi_2 = 14nz_ov^2 - 2n + (4-20n + 28ny_o)v + 14nv^2z. \qquad \text{(A-11)}$$

But for $z = 0$, from (A-9) we see that

$$va_{01} + a_{10} - b_{10}v - b_{01}v^2 = 0. \qquad \text{(A-12)}$$

If we let v_1 be a root of (A-12) and further define

$$v = V + v_1, \qquad \text{(A-13)}$$

then equation (A-9) becomes

$$z\frac{dV}{dz} = A_{01}V + A_{10}z + A_{20}z^2 + A_{11}Vz + A_{02}V^2 + \ldots \qquad \text{(A-14)}$$

and we give some of the values of the A's as follows:

$$A_{01} = \frac{a_{01} - b_{10} - 2v_1 b_{01}}{b_{10} + b_{01}v_1}, \qquad \text{(A-15)}$$

$$A_{10} = \frac{(-1-n + 3ny_o)v_1^2 - 3nv_1 - 14nz_ov_1^3}{b_{10} + b_{01}v}$$
$$- \frac{2nv_1 + (4-20n + 28ny_o)v_1^2}{b_{10} + b_{01}v}, \qquad \text{(A-16)}$$

$$A_{20} = \frac{nv_1^3 - 14nv_1^3 - \left[14nz_o v_1^2 - 2n + (4-20n + 28ny_o)v_1\right] A_{10}}{b_{10} + b_{01} v_1}, \qquad \text{(A-17)}$$

$$A_{11} = \frac{2(-1-n + 3ny_o)v_1 - 3n - 42nz_o v_1^2}{b_{10} + b_{01} v_1}$$

$$- \frac{+2nv_1 + 2(4-20n + 28ny_o)v_1}{b_{10} + b_{01} v_1}$$

$$- \frac{A_{01}\,(14nz_o v_1^2 - 2n + (4-20n + 28ny_o)v_1)}{b_{10} + b_{01} v_1}$$

$$- \frac{A_{10}\left[(4-20n)z_o + 28ny_o z_o\right]}{b_{10} + b_{01} v_1}. \qquad \text{(A-18)}$$

Equation (A-14) is of the form studied by Briot-Bouquet(18) and we shall make use of their results while studying the solutions of Equation (A-4) in the neighborhood of its singular points.

(2) Discussion of the Singular Points of the $\frac{dY}{dZ}$ Equation

From Equation (A-3) we readily find the singular points to be located in the (Y,Z) plane at

$$\left(\frac{1}{n}, 0\right),\ \left(\frac{1}{10n}, \frac{27}{100n^2}\right),\ (1,0),\ (0,0) \qquad \text{(A-19)}$$

and at two points (E,D) of the parabola $Z=(Y-1)^2$ given by the roots of

$$6nY^2 + 2(1-4n)Y + (1-n) = 0 \qquad \text{(A-20)}$$

Instead of using energy considerations to determine which of the roots of Equation (A-20) is appropriate we proceed as follows. Since the cavity is collapsing, the velocity should diminish behind the front of the spherical wave. In fact, this condition is necessary for the existence of the similarity solution and defines the domain in which the solution is valid. Analytically, this condition states that in a neighborhood behind the cavity, we must have

$$dY > dX, \tag{A-21}$$

and this should hold at the cavity wall. Due to the incorrect boundary condition imposed there, the inequality (Equation A-21) is undefined at the cavity wall. However, if we assume Z = say .05 there, rather than zero, then (Equation A-21) is defined and shows that unless we choose the larger of the roots of(Equation A-20) we will have $n < .4$ which corresponds to the incompressible case.

A criticism of this argument is that it leads to a sound speed at the cavity wall proportional to $\frac{1}{\dot{R}_H^{\,2}}$. However, it is certainly admissible in the context of Hunter's assertion of the existence of a self similar motion and from that point of view it is believed to be a more suitable argument than that based on energy considerations. Moreover, Hunter's numerical integration shows that in the neighborhood of the collapse, $\dot{R}$ tends to a constant value.

We recall that in deriving the Briot-Bouquet form (Equation A-14) we let v_1 be a root of

$$-b_{01}v^2 + v(a_{01} - b_{10}) + a_{10} = 0. \tag{A-22}$$

Here we shall show conclusively that we must, (at the point E defined by Equation(A-20), and the sonic parabola $Z=(Y-1)^2$) have the root of Equation(A-12) that is largest in absolute value.

First we consider Equation (A-20). From this we see that the coordinate Y satisfies

$$\frac{1}{4} < Y < \frac{1}{2}. \tag{A-23}$$

Hence

$$\frac{1}{4} < Z < \frac{9}{16} \tag{A-24}$$

and if we transfer the location of the singular point of the $\frac{dY}{dZ}$ equation to the origin by means of the transformation

$$\left.\begin{aligned} y &= Y - y_o \\ z &= Z - z_o \end{aligned}\right\}, \qquad \text{(A-26)}$$

then, at the origin, the inequalities (Equations A-24 and A-25) become

$$\left.\begin{aligned} \frac{1}{4} &< y_o < \frac{1}{2} \\ \frac{1}{4} &< z_o < \frac{9}{16} \end{aligned}\right\} \qquad \text{(A-27)}$$

Using arguments similar to those establishing the $dY > dX$, we have, at E,

$$dY > dZ. \qquad \text{(A-28)}$$

But given a transformation of variables of the form (Equation A-26), it is easy to show that $\frac{dY}{dZ} > 0$ implies $\frac{dy}{dz} > 0$. For if a function $t = p(q)$ undergoes a linear transformation which maps

$$t \rightarrow \bar{t}$$

and

$$q \rightarrow \bar{q}$$

such that $(t - \bar{t})$

and $(q - \bar{q})$

are of the same sign, then $\frac{dq}{dt} > 0 \rightarrow \frac{d\bar{q}}{d\bar{t}} > 0$

and conversely.

Here $\bar{t}$ and $\bar{q}$ may be identified with $y + y_o$ and $z + z_o$ respectively and these quantities are always positive. Therefore, this discussion shows that we may restrict our attention to Equation (A-4) which defines $\frac{dy}{dz}$. Now from Equation (A-12), and expressions (A-7) we see that

$$v_1 = \frac{2a_{10}}{a_{01} + b_{10} \pm \sqrt{(a_{01} + b_{10})^2 - 4(a_{01}b_{10} - a_{10}b_{01})}} - \frac{a_{10}}{a_{01}}, \qquad \text{(A-29)}$$

and that

$$v_1 = v_1(n).$$

Then, since $y = vz$,

$$\frac{dy}{dz} = v + z\frac{dv}{dz},$$

and at the point E we have

$$\frac{dy}{dz} = v_1(n) > 0. \qquad \text{(A-30)}$$

But Equation (A-7) and the fact that $.4 < n < 1$ shows that at E,

$$a_{10}\ ,\ a_{01}\ ,\ b_{10}\ ,\ \text{and}\ b_{01} \qquad \text{(A-31)}$$

are all negative. Hence if v_1 is to be positive, then

$$a_{01} + b_{10} \pm \sqrt{(a_{01} + b_{10})^2 - 4(a_{01}\, b_{10} - a_{10}\, b_{01})} < 0. \qquad \text{(A-32)}$$

But with the range assigned to the variables given by (A-31) it is easy to show, since $.4 < n < 1$, that

$$(a_{01}\, b_{10} - a_{10}\, b_{01}) < 0. \qquad \text{(A-33)}$$

Hence unless the minus sign is chosen, v_1 will be negative. This argument is conclusive whereas Hunter cannot exclude the possibility of "exceptional cases".

There are three pertinent singular points of the $\frac{dY}{dZ}$ equation, and when we transform each of them to the origin we have the following classification of the singular points as exhibited by the nature of the characteristic roots λ_1, λ_2.

At (1,0),

$$\lambda_1 = 2.667$$

$$\lambda_2 = -.448$$

and hence the cavity wall is a saddle point.

At (0,0)

$$\lambda_1 = 1$$

$$\lambda_2 = 2$$

and hence the origin is a node.

At $E(y_0 = .39428, \quad z_0 = .3669)$

$$\lambda_1 = -.92755$$

$$\lambda_2 = -.05855$$

so that E is a nodal singularity .

Hunter's paper appears to have a misprint concerning the drawing of the integral curves at the node E and we shall include the correct picture here.

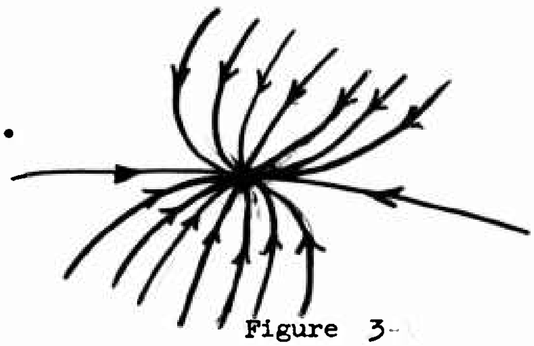

Figure 3.

INTEGRAL CURVES NEAR NODE E

(3) On the Integration of the $\frac{dY}{dZ}$ Equation.

In order to integrate the equation *

$$\frac{dZ}{dY} = \frac{2Z(-nZ + 7nY^2 + 2Y(1-5n) + 1)}{Y(Y-1)\,(nY-1) - \frac{1}{3}Z\,(9nY + n-1)}, \qquad \text{(A-34)}$$

it is necessary to investigate the behavior in the neighborhood of singular points. We get appropriate expansions directly from the theory of Briot-Bouquet. They show that if in Equation (A-14), A_{01} is not equal to a positive integer, then the Equation (A-34) has unique, regular expansions

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* Note that we integrate the inverse of $\frac{dY}{dZ}$ since Z is a single valued function of Y.

in the neighborhood of singular points. With the choice of the root v_1 at the node E, and a much simpler analysis at the other singular points, it is seen that A_{01} is never a positive integer. Then, expanding about a singular point y_o, z_o we have

$$Y - y_o = (Z - z_o)v_1 + c_1(Z - z_o)^2 + c_2(Z - z_o)^3 + \ldots \qquad \text{(A-35)}$$

where

$$c_1 = \frac{A_{10}}{1-A_{01} - A_{11}},$$

and

$$c_2 = \frac{A_{01}c_1^2 + A_{20}}{2 - A_{01} - A_{11}}. \qquad \text{(A-36)}$$

Then Equation (A-34) i.e., the $\frac{dY}{dZ}$ equation, has the following expansions.

at the origin:

$$Y = .14827Z + .008422Z^2 + .00988Z^3 + \ldots$$

at the point E:

$$Y = -.27701 + 2.166Z - .969Z^2 + \ldots \qquad \text{(A-37)}$$

at the point (Y = 1, Z = 0),

$$Y = 1 - .48766Z - .09641Z^2 + \ldots$$

With the expansions given by Equations (A-37) we can integrate Equation (A-34). This integration was carried out by a fourth order Runge-Kutta technique. The results of this integration are shown in Figure 4 and these results are tabulated in Table 2.

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TABLE 1

TABULATION OF THE FUNCTIONS $f^{(o)}$ and $g^{(o)}$

x	$f^{(o)}$	$g^{(o)}$	x	$f^{(o)}$	$g^{(o)}$	x	$f^{(o)}$	$g^{(o)}$
1	1	0	1.23	.759	.620	1.46	.615	.835
1.01	.980	.050	1.24	.752	.635	1.47	.610	.840
1.02	.967	.075	1.25	.745	.650	1.48	.605	.845
1.03	.955	.090	1.26	.737	.665	1.49	.600	.850
1.04	.945	.135	1.27	.729	.680	1.50	.595	.855
1.05	.935	.200	1.28	.721	.695	1.51	.590	.86
1.06	.925	.235	1.29	.713	.710	1.52	.585	.865
1.07	.910	.265	1.30	.705	.720	1.53	.580	.870
1.08	.900	.300	1.31	.700	.730	1.54	.575	.873
1.09	.892	.342	1.32	.690	.742	1.55	.573	.875
1.10	.880	.365	1.33	.685	.750	1.56	.570	.877
1.11	.873	.385	1.34	.680	.755	1.57	.565	.879
1.12	.863	.4140	1.35	.675	.765	1.58	.560	.881
1.13	.851	.455	1.36	.670	.775	1.59	.555	.884
1.14	.840	.470	1.37	.665	.785	1.60	.551	.885
1.15	.830	.490	1.38	.660	.790	1.61	.548	.886
1.16	.820	.510	1.39	.655	.795	1.62	.544	.887
1.17	.813	.525	1.40	.650	.800	1.63	.542	.888
1.18	.805	.540	1.41	.645	.810	1.64	.538	.889
1.19	.795	.560	1.42	.640	.815	1.65	.535	.890
1.20	.785	.575	1.43	.635	.820	1.66	.531	.891
1.21	.775	.590	1.44	.625	.825	1.67	.527	.894
1.22	.767	.607	1.45	.620	.830	1.68	.523	.895

TABLE 1 (cont.)

x	$f^{(o)}$	$g^{(o)}$	x	$f^{(o)}$	$g^{(o)}$			
1.69	.519	.895	1.93	.449	.8675	2.17	.397	.805
1.70	.515	.894	1.94	.446	.8650	2.18	.395	.803
1.71	.512	.893	1.95	.443	.8625	2.19	.394	.801
1.72	.510	.892	1.96	.440	.8600	2.20	.393	.798
1.73	.508	.891	1.97	.438	.8575	2.21	.391	.795
1.74	.505	.890	1.98	.435	.8550	2.22	.388	.792
1.75	.501	.889	1.99	.432	.8525	2.23	.385	.789
1.76	.498	.888	2.00	.430	.850	2.24	.383	.787
1.77	.495	.887	2.01	.428	.848	2.25	.380	.785
1.78	.491	.886	2.02	.427	.846	2.26	.378	.783
1.79	.488	.885	2.03	.425	.844	2.27	.376	.781
1.80	.485	.884	2.04	.423	.842	2.28	.374	.778
1.81	.482	.883	2.05	.421	.840	2.29	.372	.774
1.82	.480	.882	2.06	.419	.837	2.30	.370	.771
1.83	.478	.881	2.07	.417	.834	2.31	.368	.768
1.84	.475	.880	2.08	.414	.831	2.32	.366	.765
1.85	.472	.879	2.09	.412	.828	2.33	.364	.762
1.86	.468	.878	2.10	.410	.825	2.34	.362	.759
1.87	.465	.878	2.11	.408	.823	2.35	.360	.756
1.88	.460	.877	2.12	.407	.820	2.36	.358	.752
1.89	.458	.876	2.13	.405	.818	2.37	.356	.749
1.90	.456	.8750	2.14	.402	.814	2.38	.354	.746
1.91	.454	.8725	2.15	.400	.810	2.39	.352	.743
1.92	.451	.8700	2.16	.398	.808	2.40	.350	.740

TABLE 1 (cont.)

x	$f^{(o)}$	$g^{(o)}$
2.41	.348	.738
2.42	.346	.736
2.43	.344	.734
2.44	.343	.732
2.45	.341	.730
2.46	.340	.728
2.47	.339	.726
2.48	.337	.724
2.49	.336	.722
2.50	.335	.720
5.00	.135	.310

TABLE 2

TABULATION OF THE INTEGRATION OF EQUATION A-33

Y	Z	Y	Z	Y	Z	Y	Z
0	0	.26	.365	.52	.412	.78	.209
.01	.021	.27	.373	.53	.408	.79	.197
.02	.041	.28	.379	.54	.404	.80	.185
.03	.061	.29	.386	.55	.399	.81	.174
.04	.079	.30	.392	.56	.394	.82	.162
.05	.097	.31	.398	.57	.389	.83	.150
.06	.115	.32	.403	.58	.383	.84	.138
.07	.132	.33	.407	.59	.377	.85	.126
.08	.149	.34	.412	.60	.373	.86	.115
.09	.165	.35	.415	.61	.368	.87	.103
.10	.181	.36	.419	.62	.363	.88	.092
.11	.196	.37	.421	.63	.358	.89	.080
.12	.211	.38	.424	.64	.350	.90	.069
.13	.225	.39	.426	.65	.342	.91	.059
.14	.238	.40	.427	.66	.333	.92	.049
.15	.252	.41	.428	.67	.324	.93	.039
.16	.265	.42	.429	.68	.315	.94	.031
.17	.277	.43	.429	.69	.305	.95	.022
.18	.288	.44	.429	.70	.296	.96	.015
.19	.299	.45	.428	.71	.286	.97	.009
.20	.311	.46	.427	.72	.275	.98	.004
.21	.321	.47	.426	.73	.264	.99	.002
.22	.331	.48	.424	.74	.254	1.00	.000
.23	.339	.49	.422	.75	.243		
.24	.349	.50	.419	.76	.232		
.25	.357	.51	.416	.77	.220		

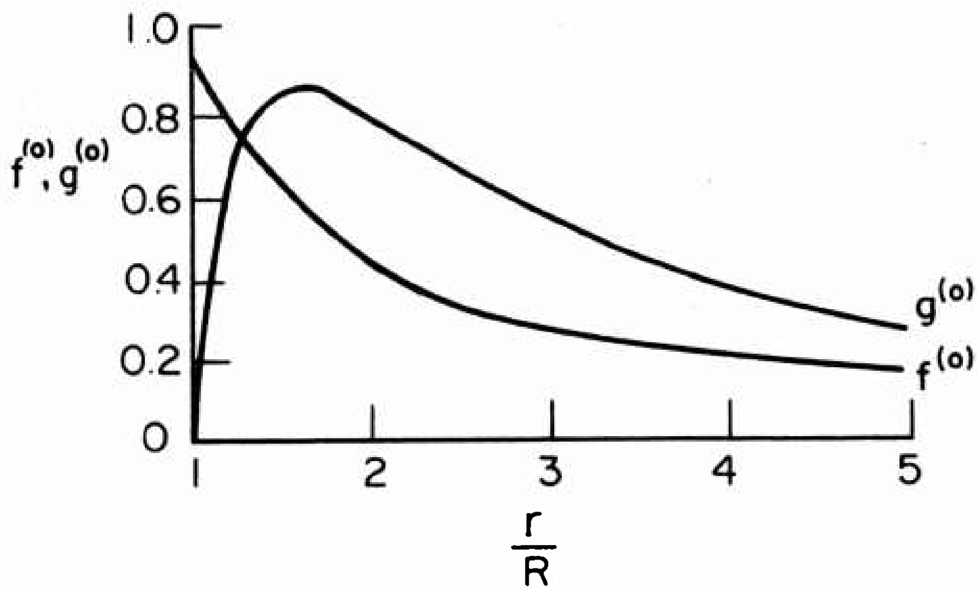

FIG. 1 THE SELF-SIMILAR SOLUTION CURVES
$n = 0.5552$

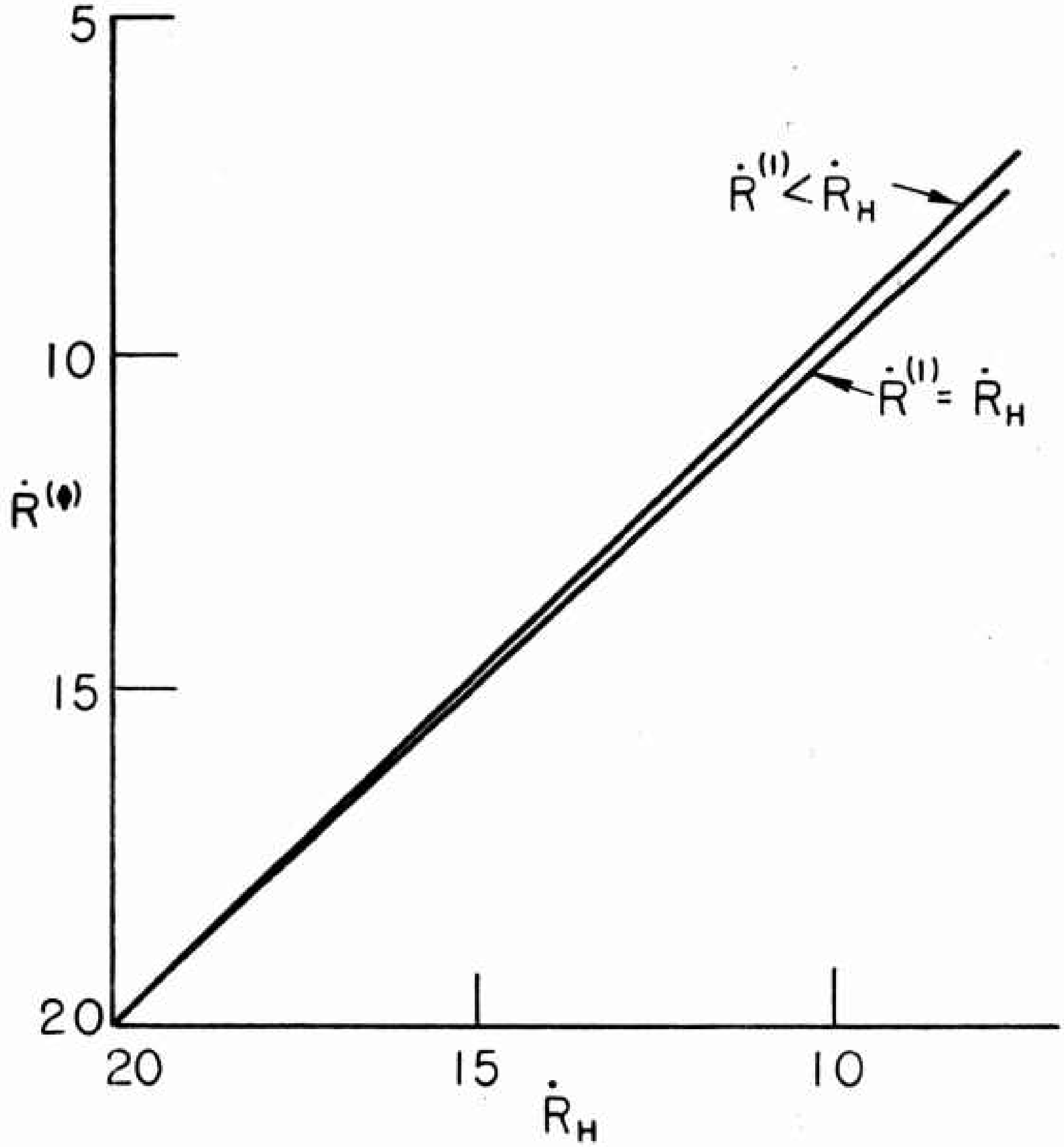

FIG. 2 COMPARISON BETWEEN $\dot{R}^{(\phi)}$ AND $\dot{R}_H$

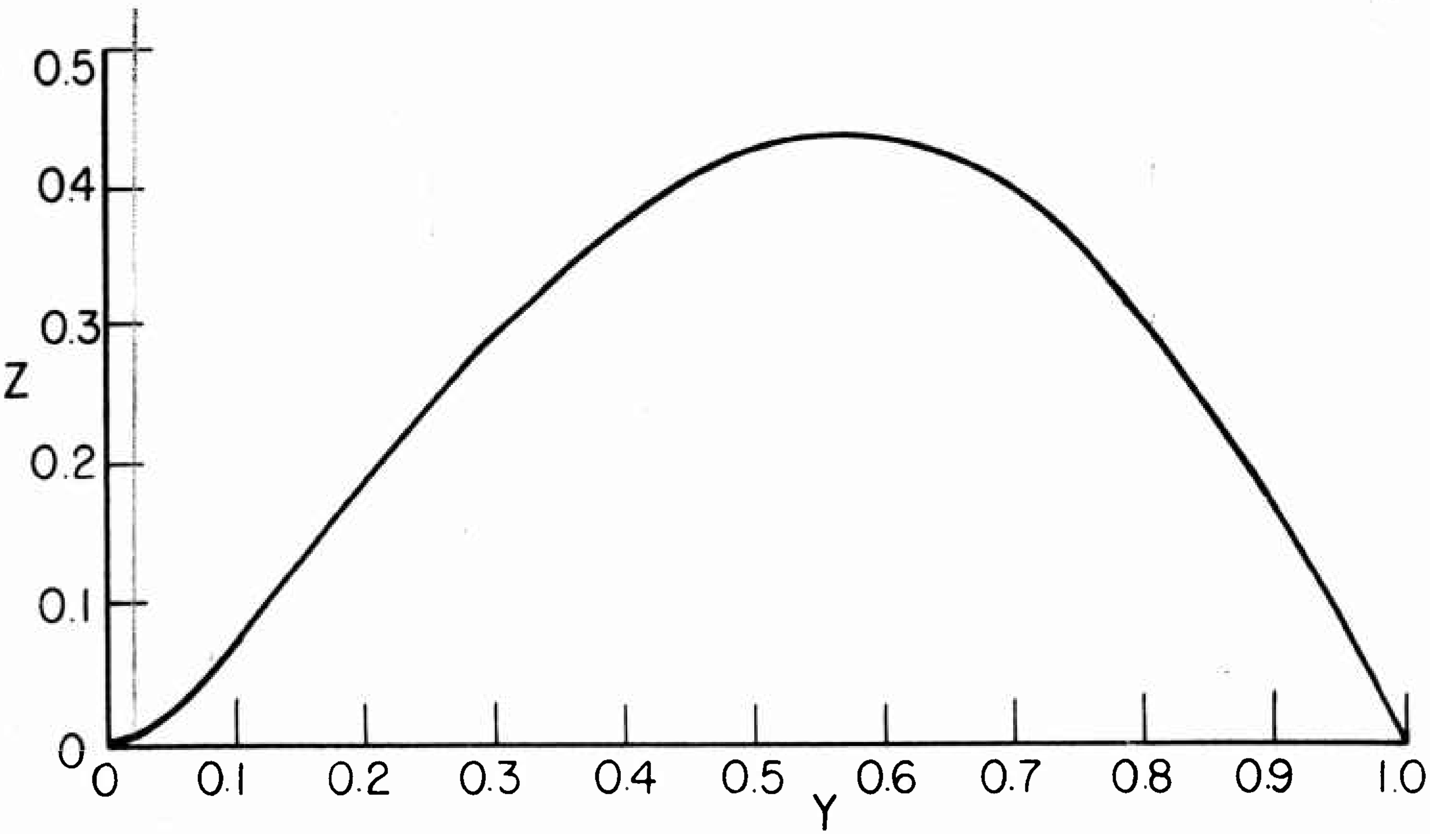

FIG. 4 INTEGRATION OF Y-Z EQUATION

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